Derived-Term Automata of Weighted Rational Expressions with Quotient Operators

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Abstract. Quotient operators have been rarely studied in the context of weighted rational expressions and automaton generation—in spite of the key role played by the quotient of words in formal language theory. To handle both left- and right-quotients we generalize an *expansion*-based construction of the *derived-term* (or *Antimirov*, or *equation*) automaton and rely on support for a *transposition* (or *reversal*) operator. The resulting automata may have spontaneous transitions, which requires different techniques from the usual derived-term constructions.

1 Introduction

There are several well-known algorithms to build an automaton from a rational expression. We are particularly interested in the construction of the *derived-term* automaton, pioneered by the *derivatives* of Brzozowski [4], improved as *partial derivatives* by Antimirov [3], and generalized to *weighted* expressions by Lombardy and Sakarovitch [13].

Thiemann [16] explores the properties of rational expression operators that enable the construction of the derived-term automaton. In particular, he shows that the left- and right- *quotients* are not "ε-testable", and that *transposition* (aka *reversal*) is neither "left nor right derivable". Our purpose is to show how *expansions* allow to overcome these issues and succeed in supporting the operators.

Our contributions include (i) a proof of the "super **S**" property, (ii) an extension of rational expressions to support transpose, left- and right-quotient operators, (iii) an algorithm to build the derived-term automaton of such an expression which requires (iv) the support of spontaneous transitions in derived-term automata.

We settle the notations and left quotient in Sect. 2. Rational expansions are introduced and computed from an expression in Sect. 3; they are used in Sect. 4 to construct the derived-term automaton. Handled in a different way, the transpose operator is introduced in Sect. 5 and used to define the right quotient. In Sect. 6 we present related work and conclude in Sect. 7.

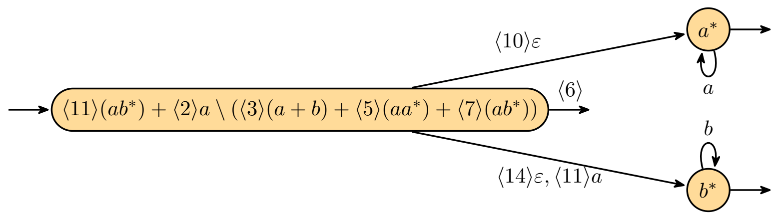


Fig. 1. The derived-term automaton of our running example,
$\mathsf{E}_1 \coloneqq (\langle 2\rangle a) \setminus (\langle 3\rangle(a+b) + \langle 5\rangle aa^* + \langle 7\rangle ab^*) + \langle 11\rangle ab^*$.

Vcsn is a free-software platform dedicated to weighted automata and rational expressions [9]. All of constructs presented in this paper can be experimented from a simple web-browser[1].

2 Notations

Our purpose is to introduce a left-quotient operator $\setminus$ for weighted rational expressions (e.g., $\mathsf{E}_1 \coloneqq (\langle 2\rangle a) \setminus (\langle 3\rangle(a+b) + \langle 5\rangle(aa^*) + \langle 7\rangle(ab^*)) + \langle 11\rangle(ab^*)$, weights are in angle brackets), and to build an equivalent automaton from it (Fig. 1). To this end we compute *the rational expansion* of an expression [7]:

$$d(\mathsf{E}_1) = \underbrace{\underbrace{\overbrace{\varepsilon}^{\text{Label}}}_{\text{First}} \odot \overbrace{\Big[\underbrace{\overbrace{\langle 6\rangle}^{\text{Weight}}}_{\text{Immediate Constant term}} \odot 1 \oplus \langle 10\rangle \odot \overbrace{a^*}^{\text{Expression (Sect. 2.2)}} \oplus \overbrace{\langle 14\rangle \odot \underbrace{b^*}_{\text{Derived term}}}^{\text{Monomial}}\Big]}^{\text{Polynomial (Sect. 2.3)}} \oplus \underbrace{\overbrace{a}^{\text{Label}}}_{\text{First}} \odot \Big[\overbrace{\langle 11\rangle \odot b^*}^{\text{Polynomial}}\Big]}_{\text{Expansion (Sect. 3.1)}}$$

Expansions can be thought as a (non unique) normal form for expressions. Defining them requires several concepts, introduced bottom-up in this section.

2.1 Rational Series

Series are to weighted automata what languages are to Boolean automata. Not all languages are rational (denoted by an expression), and similarly, not all series are rational (denoted by a weighted expression). We follow Sakarovitch [15].

Let A be a (finite) alphabet; a *word* m is a finite sequence of letters of A. The empty word is denoted ε. The set of words is written A^*, and $A^?$ denotes $A \cup \{\varepsilon\}$. A *language* is a subset of A^*. Let $\langle \mathbb{K}, +, \cdot, 0_{\mathbb{K}}, 1_{\mathbb{K}}\rangle$ be a commutative semiring whose multiplication will be denoted by implicit concatenation. A (formal power) *series* over A^* with *weights* (or *multiplicities*) in $\mathbb{K}$ is any map from A^* to $\mathbb{K}$. The weight of a word m in a series s is denoted $s(m)$. The *empty* series, $m \mapsto 0_{\mathbb{K}}$,

[1] See the interactive environment, `http://vcsn-sandbox.lrde.epita.fr`, or the companion notebook, `http://vcsn.lrde.epita.fr/dload/doc/ICTAC-2017.html`.

is denoted 0; for any word u (including ε), u denotes the series $m \mapsto 1_{\mathbb{K}}$ if $m = u$, $0_{\mathbb{K}}$ otherwise. Equipped with the pointwise addition ($s + t \coloneqq m \mapsto s(m) + t(m)$) and the Cauchy product ($s \cdot t \coloneqq m \mapsto \sum_{u,v \in A^* | u \cdot v = m} s(u) \cdot t(v)$) as multiplication, the set of these series forms a semiring denoted $\langle \mathbb{K}\langle\langle A^* \rangle\rangle, +, \cdot, 0, \varepsilon \rangle$.

The *constant term* of a series s, denoted s_ε, is $s(\varepsilon)$, the weight of the empty word. A series s is *proper* if $s_\varepsilon = 0_{\mathbb{K}}$. The *proper part* of s, denoted s_p, is the proper series which coincides with s on non empty words: $s = s_\varepsilon \varepsilon + s_p$ (or, with a slight abuse of notations $s = s_\varepsilon + s_p$).

Star. A weight $k \in \mathbb{K}$ is *starrable* if its *star*, $k^* \coloneqq \sum_{n \in \mathbb{N}} k^n$, is defined. We suppose that $\mathbb{K}$ is a *topological semiring*, i.e., it is equipped with a topology, and both addition and multiplication are continuous. Besides, it is supposed to be *strong*, i.e., the product of two summable families is summable. This ensures that $\mathbb{K}\langle\langle A^* \rangle\rangle$, equipped with the product topology derived from the topology on $\mathbb{K}$, is also a strong topological semiring. The *star* of a series is an infinite sum: $s^* \coloneqq \sum_{n \in \mathbb{N}} s^n$.

To prove the correctness of our construct (Proposition 6), we will need a property of star (Proposition 2) which follows from the following result. In various forms it is named the "denesting rule" [11, p. 57], the "property **S**" [15, Propositions III.2.5 and III.2.6], or the "sum-star equation" [10, p. 188]. Proofs can be found for the axiomatic approach of star (based on Conway semirings), but we followed the topology-based one, for which we did not find a published version.

Proposition 1 (Super S). *Let $\mathbb{K}$ be a strong topological semiring. For any series $s, t \in \mathbb{K}\langle\langle A^* \rangle\rangle$, if s_ε^*, $(t_\varepsilon s_\varepsilon^*)^*$, and $(s_\varepsilon + t_\varepsilon)^*$ are defined and $(s_\varepsilon + t_\varepsilon)^* = s_\varepsilon^*(t_\varepsilon s_\varepsilon^*)^*$, then $(s + t)^* = s^*(ts^*)^*$.*

Proof. This proof climbs from restricted forms (e.g., s being a weight and t being proper) to the general cases using previous steps. See Appendix A.1. ☐

All the usual semirings ($\mathbb{Q}, \mathbb{R}, \mathbb{R}_{\min}, \mathrm{Log}$, etc.) are strong topological semirings, in which if s_ε^*, $(t_\varepsilon s_\varepsilon^*)^*$, and $(s_\varepsilon + t_\varepsilon)^*$ are defined then $(s_\varepsilon + t_\varepsilon)^* = s_\varepsilon^*(t_\varepsilon s_\varepsilon^*)^*$. Proposition 1 (and Proposition 2) actually do not need $\mathbb{K}$ to be commutative.

Proposition 2. *Let $\mathbb{K}$ be a strong topological semiring. Let $s \in \mathbb{K}, t \in \mathbb{K}\langle\langle A^* \rangle\rangle$, if s^*, $(t_\varepsilon s^*)^*$, and $(s + t_\varepsilon)^*$ are defined and $(s + t_\varepsilon)^* = s^*(t_\varepsilon s^*)^*$ then $(s + t)^* = s^* + s^* t (s + t)^*$.*

Proof. The result follows from Proposition 1, and from $(ts^*)^* = \varepsilon + (ts^*)(ts^*)^*$: $(s+t)^* = s^*(ts^*)^* = s^*(\varepsilon + (ts^*)(ts^*)^*) = s^* + s^* t(s^*(ts^*)^*) = s^* + s^* t(s+t)^*$. ☐

Left Quotient. Like Li et al. [12], we define the left quotient by series s of series t as: $s \setminus t \coloneqq v \mapsto \sum_{u \in A^*} s(u) \cdot t(uv)$.

Proposition 3 (Quotient is bilinear [12, Proposition 6]).
For weight $k \in \mathbb{K}$ and series $s, s', t, t' \in \mathbb{K}\langle\langle A^* \rangle\rangle$:

$$s \setminus (t + t') = s \setminus t + s \setminus t' \qquad s \setminus kt = k(s \setminus t)$$
$$(s + s') \setminus t = s \setminus t + s' \setminus t \qquad (ks) \setminus t = k(s \setminus t)$$

Let u, v be two words, their *root* $r(u, v)$ is u if u is a prefix of v, v if v is a prefix of u, undefined otherwise.

Proposition 4. *For series $s, t \in \mathbb{K}\langle\langle A^*\rangle\rangle$ and words $u, v \in A^*$:*

$$us \setminus vt = \begin{cases} 0 & \text{if } r(u,v) \text{ is undefined} \\ u's \setminus v't & \text{otherwise, with } u' = r(u,v) \setminus u, v' = r(u,v) \setminus v \end{cases}$$

2.2 Extended Weighted Rational Expressions

Definition 1 (Extended Weighted Rational Expression). *A* rational expression E *is a term built from the following grammar, where $a \in A$ is a letter, and $k \in \mathbb{K}$ a weight:* $\mathsf{E} ::= \mathsf{0} \mid \mathsf{1} \mid a \mid \mathsf{E} + \mathsf{E} \mid \langle k\rangle\mathsf{E} \mid \mathsf{E} \cdot \mathsf{E} \mid \mathsf{E}^* \mid \mathsf{E} \setminus \mathsf{E}$.

Example 1. Let $\mathsf{E}_1 \coloneqq (\langle 2\rangle a) \setminus (\langle 3\rangle(a+b) + \langle 5\rangle aa^* + \langle 7\rangle ab^*) + \langle 11\rangle ab^*$. By "simplifying" the left quotient (distributivity and $(\langle 2\rangle a) \setminus (\langle 3\rangle(a+b)) \equiv \langle 6\rangle\mathsf{1}$, etc.), it can be shown to be equivalent to $\langle 6\rangle\mathsf{1} + \langle 10\rangle a^* + \langle 14\rangle b^* + \langle 11\rangle ab^*$.

Rational expressions are syntactic objects; they provide a finite notation for (some) series, which are semantic objects.

Definition 2 (Series Denoted by an Expression). *Let E be an expression. The series denoted by E, noted $[\![\mathsf{E}]\!]$, is defined by induction on E:*

$$[\![\mathsf{0}]\!] \coloneqq 0 \qquad [\![\mathsf{1}]\!] \coloneqq \varepsilon \qquad [\![a]\!] \coloneqq a \qquad [\![\mathsf{E}+\mathsf{F}]\!] \coloneqq [\![\mathsf{E}]\!] + [\![\mathsf{F}]\!] \qquad [\![\langle k\rangle\mathsf{E}]\!] \coloneqq k[\![\mathsf{E}]\!]$$

$$[\![\mathsf{E}\cdot\mathsf{F}]\!] \coloneqq [\![\mathsf{E}]\!] \cdot [\![\mathsf{F}]\!] \qquad [\![\mathsf{E}^*]\!] \coloneqq [\![\mathsf{E}]\!]^* \qquad [\![\mathsf{E}\setminus\mathsf{F}]\!] \coloneqq [\![\mathsf{E}]\!] \setminus [\![\mathsf{F}]\!]$$

An expression is *valid* if it denotes a series. More specifically, this requires that $[\![\mathsf{F}]\!]^*$ is well defined for each sub-expression of the form F^*, i.e., that the constant term of $[\![\mathsf{F}]\!]$ is *starrable* in $\mathbb{K}$ (Proposition 2). So for instance, $1_{\mathbb{K}}^*$ and $(a^*)^*$ are valid in $\mathbb{B}$, but invalid in $\mathbb{Q}$.

Two expressions E and F are *equivalent* iff $[\![\mathsf{E}]\!] = [\![\mathsf{F}]\!]$. Some expressions are "trivially equivalent"; any candidate expression will be rewritten via the following *trivial identities*. Any sub-expression of a form listed to the left of a '$\Rightarrow$' is rewritten as indicated on the right.

$$\mathsf{E}+\mathsf{0} \Rightarrow \mathsf{E} \qquad \mathsf{0}+\mathsf{E} \Rightarrow \mathsf{E}$$

$$\langle 0_{\mathbb{K}}\rangle\mathsf{E} \Rightarrow \mathsf{0} \quad \langle 1_{\mathbb{K}}\rangle\mathsf{E} \Rightarrow \mathsf{E} \quad \langle k\rangle\mathsf{0} \Rightarrow \mathsf{0} \quad \langle k\rangle\langle h\rangle\mathsf{E} \Rightarrow \langle kh\rangle\mathsf{E}$$

$$(\langle k\rangle^{?}\mathsf{1}) \cdot \mathsf{E} \Rightarrow \langle k\rangle\mathsf{E} \quad \mathsf{E} \cdot (\langle k\rangle^{?}\mathsf{1}) \Rightarrow \langle k\rangle\mathsf{E}$$

$$\mathsf{E}\cdot\mathsf{0} \Rightarrow \mathsf{0} \qquad \mathsf{0}\cdot\mathsf{E} \Rightarrow \mathsf{0} \qquad \mathsf{0}^{\star} \Rightarrow \mathsf{1} \qquad \mathsf{0}\setminus\mathsf{E} \Rightarrow \mathsf{0} \qquad \mathsf{E}\setminus\mathsf{0} \Rightarrow \mathsf{0} \qquad \mathsf{1}\setminus\mathsf{E} \Rightarrow \mathsf{E}$$

where E stands for a rational expression, $\ell \in A^?$ is a *label*, $k, h \in \mathbb{K}$ are weights, and $\langle k\rangle^{?}\ell$ denotes either $\langle k\rangle\ell$, or ℓ in which case $k = 1_{\mathbb{K}}$ in the right-hand side of $\Rightarrow$. The choice of these identities is beyond the scope of this paper [13, p. 149], they are limited to trivial properties; in particular *linearity* ("weighted ACI": associativity, commutativity, and $\langle k\rangle^{?}\mathsf{E} + \langle h\rangle^{?}\mathsf{E} \Rightarrow \langle k+h\rangle\mathsf{E}$) is not enforced — polynomials will take care of it (Sect. 2.3). In practice, additional identities help reducing the number of derived terms, hence the final automaton size.

2.3 Rational Polynomials

The "partial derivatives" [3] rely on *sets* of rational expressions, later generalized to *weighted sets* [13], i.e., functions (partial, with finite domain) from the set of expressions into $\mathbb{K} \setminus \{0_{\mathbb{K}}\}$. It proves useful to view such structures as *polynomials* of rational expressions. In essence, they capture the linearity of addition.

Definition 3 (Rational Polynomial). *A* polynomial *(of rational expressions) is a finite (left) linear combination of rational expressions. Syntactically it is represented by a term built from the grammar* $\mathsf{P} ::= 0 \mid \langle k_1\rangle \odot \mathsf{E}_1 \oplus \cdots \oplus \langle k_n\rangle \odot \mathsf{E}_n$ *where* $k_i \in \mathbb{K} \setminus \{0_{\mathbb{K}}\}$ *denote* non-zero *weights, and* E_i *denote* non-zero *expressions. Expressions may not appear more than once in a polynomial. A* monomial *is a pair* $\langle k_i\rangle \odot \mathsf{E}_i$*. The* terms *of* P *is the set* $\mathsf{exprs}(\mathsf{P}) := \{\mathsf{E}_1, \ldots, \mathsf{E}_n\}$*.*

We use specific symbols ($\odot$ and $\oplus$) to clearly separate the outer polynomial layer from the inner expression layer. A polynomial P of expressions can be "projected" as a rational expression $\mathsf{expr}(\mathsf{P})$ by mapping its sum and left multiplication by a weight onto the corresponding operators on rational expressions. This operation is performed on a canonical form of the polynomial (expressions are sorted in a well defined order). Polynomials denote series: $[\![\mathsf{P}]\!] := [\![\mathsf{expr}(\mathsf{P})]\!]$.

Example 2 (Example 1 continued). Let $\mathsf{E}_1 := (\langle 2\rangle a) \setminus (\langle 3\rangle(a+b) + \langle 5\rangle aa^* + \langle 7\rangle ab^*) + \langle 11\rangle ab^*$. The polynomial '$\mathsf{P}_{1\varepsilon} := \langle 6\rangle \odot 1 \oplus \langle 10\rangle \odot a^* \oplus \langle 14\rangle \odot b^*$' has three monomials, and $\mathsf{expr}(\mathsf{P}_{1\varepsilon}) = \langle 6\rangle 1 + \langle 10\rangle a^* + \langle 14\rangle b^*$.

Let $\ell \in A^?$ be a label, $\mathsf{P} = \langle k_1\rangle \odot \mathsf{E}_1 \oplus \cdots \oplus \langle k_n\rangle \odot \mathsf{E}_n$ a polynomial, k a weight (possibly zero) and F an expression (possibly zero), we introduce:

$$\begin{aligned}
\ell \cdot \mathsf{P} &:= \langle k_1\rangle \odot (\ell \cdot \mathsf{E}_1) \oplus \cdots \oplus \langle k_n\rangle \odot (\ell \cdot \mathsf{E}_n)\\
\mathsf{P} \cdot \mathsf{F} &:= \langle k_1\rangle \odot (\mathsf{E}_1 \cdot \mathsf{F}) \oplus \cdots \oplus \langle k_n\rangle \odot (\mathsf{E}_n \cdot \mathsf{F})\\
\langle k\rangle \mathsf{P} &:= \langle kk_1\rangle \odot \mathsf{E}_1 \oplus \cdots \oplus \langle kk_n\rangle \odot \mathsf{E}_n\\
\mathsf{P}_1 \setminus \mathsf{P}_2 &:= \bigoplus_{\substack{\langle k_1\rangle \odot \mathsf{E}_1 \in \mathsf{P}_1\\ \langle k_2\rangle \odot \mathsf{E}_2 \in \mathsf{P}_2}} \langle k_1 \cdot k_2\rangle \odot (\mathsf{E}_1 \setminus \mathsf{E}_2)
\end{aligned} \tag{1}$$

Trivial identities might simplify the result, e.g., $(\langle 1_{\mathbb{K}}\rangle \odot 1) \setminus (\langle 1_{\mathbb{K}}\rangle \odot a) = \langle 1_{\mathbb{K}}\rangle \odot a$. Note the asymmetry between left and right exterior products. Addition is commutative, multiplication by zero (be it an expression or a weight) evaluates to the polynomial zero, and left multiplication by a weight is distributive.

Lemma 1. $[\![\ell \cdot \mathsf{P}]\!] = \ell \cdot [\![\mathsf{P}]\!]$ $\quad [\![\mathsf{P} \cdot \mathsf{F}]\!] = [\![\mathsf{P}]\!] \cdot [\![\mathsf{F}]\!]$
$[\![\langle k\rangle \mathsf{P}]\!] = \langle k\rangle [\![\mathsf{P}]\!]$ $\quad [\![\mathsf{P}_1 \setminus \mathsf{P}_2]\!] = [\![\mathsf{P}_1]\!] \setminus [\![\mathsf{P}_2]\!]$.

Proof. These properties are trivial. In particular, the case of $\setminus$ follows from Proposition 3 (see Appendix A.2). ◻

2.4 Weighted Automata

Definition 4. *A finite* weighted automaton $\mathcal{A}$ *is a tuple* $\langle A, \mathbb{K}, Q, E, I, T\rangle$ *where:*
- A *is an alphabet,*
- $\mathbb{K}$ *(the set of weights) is a semiring,*
- Q *is a finite set of states,*
- I *and* T *are the* initial *and* final *functions from* Q *into* $\mathbb{K}$*,*
- E *is a (partial) function from* $Q \times A^? \times Q$ *into* $\mathbb{K} \setminus \{0_\mathbb{K}\}$*;*
  its domain represents the transitions*:* $(source, label, destination)$*.*

Our automata are "ε-NFAs": they may have spontaneous transitions ($\ell \in A^?$). A *path* π is a sequence of transitions $(q_0, \ell_1, q_1)(q_1, \ell_2, q_2) \cdots (q_{n-1}, \ell_n, q_n)$ where the source of each is the destination of the previous one; its *source* is $\iota(\pi) \coloneqq q_0$, its *destination* is $\tau(\pi) \coloneqq q_n$, its *label* is the word $\ell(\pi) \coloneqq \ell_1 \cdots \ell_n$, its *weight* is $w(\pi) \coloneqq E(q_0, \ell_1, q_1) \cdot \ldots \cdot E(q_{n-1}, \ell_n, q_n)$, and its *weighted label* [14] is the monomial $wl(\pi) \coloneqq w(\pi)\ell(\pi)$. The set of paths of $\mathcal{A}$ is denoted $\mathsf{Path}(\mathcal{A})$. A *computation* c is a path π together with its initial and final functions at the ends: $c \coloneqq (I(\iota(\pi)), \pi, T(\tau(\pi)))$, its weight is $w(c) \coloneqq I(\iota(\pi))w(\pi)T(\tau(\pi))$.

The *evaluation* of word u by an automaton $\mathcal{A}$, $\mathcal{A}(u)$, is the sum of the weights of all the computations labeled by u, or $0_\mathbb{K}$ if there are none. The *behavior* of $\mathcal{A}$ is the series $[\![\mathcal{A}]\!] \coloneqq u \mapsto \mathcal{A}(u)$. A state q is *initial* if $I(q) \neq 0_\mathbb{K}$. A state q is *accessible* if there is a path from an initial state to q. The *accessible* part of an automaton $\mathcal{A}$ is the sub-automaton whose states are the accessible states of $\mathcal{A}$.

Automata with spontaneous transitions may be *invalid*, if they have cycles of spontaneous transitions whose weight is not starrable [14].

Definition 5 (Semantics of a State). *Given a weighted automaton* $\mathcal{A} = \langle A, \mathbb{K}, Q, E, I, T\rangle$*, the* semantics of state q *(aka, its* future*) is the series:*

$$[\![q]\!] \coloneqq T(q) + \sum_{\pi \in \mathsf{Path}(\mathcal{A}) \mid q = i(\pi)} wl(\pi)T(\tau(\pi)) \tag{2}$$

Clearly, $[\![\mathcal{A}]\!] = \sum_{q \in Q} I(q)[\![q]\!]$.

Proposition 5. *For any automaton* $\mathcal{A}$*, we have:*

$$[\![q]\!] = T(q) + \sum_{\ell \in A^?, q' \in Q} E(q, \ell, q')\ell[\![q']\!] \tag{3}$$

The equivalence of (2) and (3) can be seen as two different strategies of evaluation: the first one is by depth first (follow each path individually, then sum their weights), the second one by breadth (starting from the set of initial states, descend "simultaneously" each transition, and repeat).

A simple proof by induction [7, Sec. 2.5] suffices in the absence of spontaneous transitions. With cycles of spontaneous transitions, we face infinite sums whose formal treatment requires arguments that go way beyond the scope of this paper. This is in fact the core of the work of Lombardy and Sakarovitch [14].

3 Rational Expansions

Expansions (Sect. 3.1) can be viewed as a normal form of rational expansions from which the construction of the derived-term automaton is straightforward. For instance, *the* (see Sect. 3.2) expansion of $\langle 2\rangle ac+\langle 3\rangle bc$ is $a\odot[\langle 2\rangle\odot c]\oplus b\odot[\langle 3\rangle\odot c]$.

3.1 Rational Expansions

An *expansion* [7, 6] is a syntactic object that denotes a linear form of a series/expressions: it maps each label to a polynomial. From systems of expansions, building the "equation" automaton is straightforward (Sect. 4). Although closely related to the derivatives of an expression, expansions can cope more easily with new operators (such as quotient) than derivatives [6]. They also have a more "forward" flavor: their computation follow very simple rules such as distributivity. Let $[n]$ denote $\{1,\dots,n\}$.

Definition 6 (Rational Expansion). *A* rational expansion X *is a term built from the grammar* $\mathsf{X} ::= 0 \mid \ell_1\odot[\mathsf{P}_1]\oplus\cdots\oplus\ell_n\odot[\mathsf{P}_n]$ *where* $\ell_i\in A^?$ *are labels (occurring at most once), and* P_i *non-zero polynomials. The* firsts *of* X *is* $f(\mathsf{X}) \coloneqq \{\ell_1,\dots,\ell_n\}$ *(possibly empty), and its* terms *are* $\mathsf{exprs}(\mathsf{X}) \coloneqq \bigcup_{i\in[n]}\mathsf{exprs}(\mathsf{P}_i)$.

Polynomials are written in square brackets to ease reading. Given an expansion X, we denote by X_ℓ (or $\mathsf{X}(\ell)$) the polynomial corresponding to ℓ in X, or the polynomial zero if $\ell\notin f(\mathsf{X})$. Expansions will thus be written: $\mathsf{X}=\bigoplus_{\ell\in f(\mathsf{X})}\ell\odot[\mathsf{X}_\ell]$.

An expansion X can be "projected" as a rational expression $\mathsf{expr}(\mathsf{X})$ by mapping labels and polynomials to their corresponding rational expressions, and $\oplus/\odot$ to the sum/concatenation of rational expressions. Again, this is performed on a canonical form of the expansion: labels and polynomials are sorted. Expansions also denote series: $[\![\mathsf{X}]\!] \coloneqq [\![\mathsf{expr}(\mathsf{X})]\!]$. An expansion X is said to be *equivalent* to an expression E iff $[\![\mathsf{X}]\!]=[\![\mathsf{E}]\!]$.

The *immediate constant term* of an expansion X, $\mathsf{X}_\$$, is the weight of $\mathsf{1}$ in $\mathsf{X}(\varepsilon)$, or $0_{\mathbb{K}}$ if it does not exist. The *immediate proper part* of X, X_p, is the expansion which coincides with X but with a null immediate constant term; hence[2] $\mathsf{X}=\varepsilon\odot[\langle\mathsf{X}_\$\rangle\odot\mathsf{1}]\oplus\mathsf{X}_p$. Beware that $[\![\mathsf{X}_p]\!]$ might not be proper; e.g., with $\mathsf{X}\coloneqq\varepsilon\odot[\langle 2\rangle\odot\mathsf{1}\oplus\langle 3\rangle\odot a\setminus a]$, we have $\mathsf{X}_p=\varepsilon\odot[\langle 3\rangle\odot a\setminus a]$, yet $[\![\mathsf{X}_p]\!]=3$.

Example 3 (Examples 1 and 2 continued). Let $\mathsf{P}_{1a}\coloneqq\langle 11\rangle\odot b^*$. Expansion $\mathsf{X}_1\coloneqq\varepsilon\odot\mathsf{P}_{1\varepsilon}\oplus a\odot\mathsf{P}_{1a}=\varepsilon\odot[\langle 6\rangle\odot\mathsf{1}\oplus\langle 10\rangle\odot a^*\oplus\langle 14\rangle\odot b^*]\oplus a\odot[\langle 11\rangle\odot b^*]$ maps the label ε (resp. a) to the polynomial $\mathsf{P}_{1\varepsilon}$ (resp. P_{1a}). The immediate constant term of X_1 is 6. X_1 is equivalent to E_1.

Let X,Y be expansions, k a weight, and E an expression (all possibly zero):

$$\mathsf{X}\oplus\mathsf{Y}\coloneqq\bigoplus_{\ell\in f(\mathsf{X})\cup f(\mathsf{Y})}\ell\odot[\mathsf{X}_\ell\oplus\mathsf{Y}_\ell]\qquad\langle k\rangle\mathsf{X}\coloneqq\bigoplus_{\ell\in f(\mathsf{X})}\ell\odot[\langle k\rangle\mathsf{X}_\ell]$$

[2] The (straightforward) definition of addition of expansions, $\oplus$, will be given below.

$$\mathsf{X} \cdot \mathsf{E} \coloneqq \bigoplus_{\ell \in f(\mathsf{X})} \ell \odot [\mathsf{X}_\ell \cdot \mathsf{E}]$$

$$\mathsf{X} \setminus \mathsf{Y} \coloneqq \bigoplus \begin{cases} \varepsilon \odot [\mathsf{X}_\ell \setminus \mathsf{Y}_\ell] & \forall \ell \in f(\mathsf{X}) \cap f(\mathsf{Y}) \\ \varepsilon \odot [\mathsf{X}_\varepsilon \setminus (\ell' \cdot \mathsf{Y}_{\ell'})] & \forall \ell' \in f(\mathsf{Y}) \quad \text{if } \varepsilon \in f(\mathsf{X}) \\ \varepsilon \odot [(\ell \cdot \mathsf{X}_\ell) \setminus \mathsf{Y}_\varepsilon] & \forall \ell \in f(\mathsf{X}) \quad \text{if } \varepsilon \in f(\mathsf{Y}) \end{cases} \tag{4}$$

Since by definition expansions never map to null polynomials, some firsts might be smaller sets than suggested by these equations. For instance in $\mathbb{Z}$ the sum of $\varepsilon \odot [\langle 1 \rangle \odot 1] \oplus a \odot [\langle 1 \rangle \odot b]$ and $\varepsilon \odot [\langle 1 \rangle \odot 1] \oplus a \odot [\langle -1 \rangle \odot b]$ is $\varepsilon \odot [\langle 2 \rangle \odot 1]$.

With the convention that terms with undefined roots are ignored (i.e., equal to 0), the definition (4) can be stated as

$$\mathsf{X} \setminus \mathsf{Y} = \bigoplus_{\substack{\ell \in f(\mathsf{X}), \ell' \in f(\mathsf{Y}) \\ p = r(\ell, \ell')}} \varepsilon \odot \Big[((p \setminus \ell) \cdot \mathsf{X}_\ell) \setminus ((p \setminus \ell') \cdot \mathsf{Y}_{\ell'})\Big] \tag{5}$$

The following lemma is simple to establish: lift semantic equivalences, such as those of Propositions 3 and 4, to syntax, using Lemma 1 (Appendix A.3).

Lemma 2. $[\![\mathsf{X} \oplus \mathsf{Y}]\!] = [\![\mathsf{X}]\!] + [\![\mathsf{Y}]\!] \qquad [\![\langle k \rangle \mathsf{X}]\!] = \langle k \rangle [\![\mathsf{X}]\!]$
$[\![\mathsf{X} \cdot \mathsf{E}]\!] = [\![\mathsf{X}]\!] \cdot [\![\mathsf{E}]\!] \qquad [\![\mathsf{X} \setminus \mathsf{Y}]\!] = [\![\mathsf{X}]\!] \setminus [\![\mathsf{Y}]\!]$.

3.2 Expansion of a Rational Expression

Definition 7 (Expansion of a Rational Expression). *The* expansion of a rational expression E*, written* $d(\mathsf{E})$*, is defined inductively as follows:*

$$d(0) \coloneqq 0 \qquad d(1) \coloneqq \varepsilon \odot [\langle 1_{\mathbb{K}} \rangle \odot 1] \qquad d(a) \coloneqq a \odot [\langle 1_{\mathbb{K}} \rangle \odot 1]$$
$$d(\mathsf{E} + \mathsf{F}) \coloneqq d(\mathsf{E}) \oplus d(\mathsf{F}) \qquad d(\langle k \rangle \mathsf{E}) \coloneqq \langle k \rangle d(\mathsf{E})$$
$$d(\mathsf{E} \cdot \mathsf{F}) \coloneqq d_p(\mathsf{E}) \cdot \mathsf{F} \oplus \big\langle d_\$(\mathsf{E}) \big\rangle d(\mathsf{F})$$
$$d(\mathsf{E}^*) \coloneqq \varepsilon \odot \big[\big\langle d_\$(\mathsf{E})^* \big\rangle \odot 1\big] \oplus \big\langle d_\$(\mathsf{E})^* \big\rangle d_p(\mathsf{E}) \cdot \mathsf{E}^* \tag{6}$$
$$d(\mathsf{E} \setminus \mathsf{F}) \coloneqq d(\mathsf{E}) \setminus d(\mathsf{F}) \tag{7}$$

where $d_\$(\mathsf{E})$ *and* $d_p(\mathsf{E})$ *are the immediate constant term/immediate proper part of* $d(\mathsf{E})$*.*

The right-hand sides are indeed expansions. The computation trivially terminates: induction is performed on strictly smaller sub-expressions.

Proposition 6. *An expression is equivalent to its expansion.*

Proof. Follows from a straightforward induction on E [7]. For instance, the case of left quotient follows from $\big[\!\big[d(\mathsf{E} \setminus \mathsf{F})\big]\!\big] = \big[\!\big[d(\mathsf{E}) \setminus d(\mathsf{F})\big]\!\big]$(by definition (7)) $= \big[\!\big[d(\mathsf{E})\big]\!\big] \setminus \big[\!\big[d(\mathsf{F})\big]\!\big]$(by Lemma 2). The case of star is more delicate than in our previous work [7] as $d_p(\mathsf{E})$ might not denote a proper series. This is handled by Proposition 2, much more powerful than its predecessor [7, Proposition 2]. ◻

4 Expansion-Based Derived-Term Automaton

Definition 8 (Expansion-Based Derived-Term Automaton). *The* derived-term automaton *of an expression* E *over* G *is the* accessible part *of the automaton* $\mathcal{A}_\mathsf{E} \coloneqq \langle M, G, \mathbb{K}, Q, E, I, T\rangle$ *defined as follows:*
- Q *is the set of rational expressions on alphabet* A *with weights in* $\mathbb{K}$,
- $I = \mathsf{E} \mapsto 1_\mathbb{K}$,
- $E(\mathsf{F}, \ell, \mathsf{F}') = k$ *iff* $\ell \in f(d(\mathsf{F}))$ and $\langle k\rangle \odot \mathsf{F}' \in d_p(\mathsf{F})(\ell)$,
- $T(\mathsf{F}) = d_\$(\mathsf{F})$.

It is straightforward to extract an algorithm from Definition 8, using a worklist of states whose outgoing transitions need to be computed [7, Algorithm 1]. However, we must justify Definition 8 by proving that this automaton is finite.

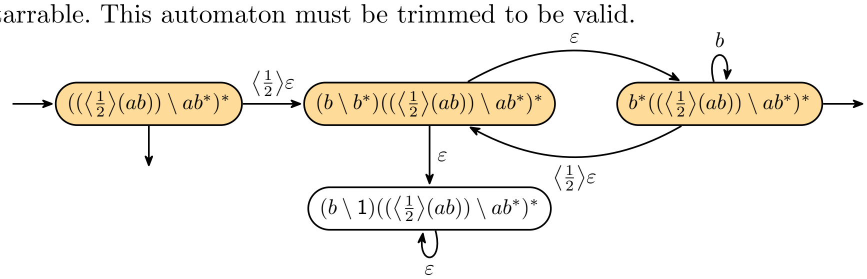
Example 4 (Examples 1 to 3 continued). With $\mathsf{E}_1 \coloneqq (\langle 2\rangle a)\backslash(\langle 3\rangle(a+b)+\langle 5\rangle aa^*+\langle 7\rangle ab^*)+\langle 11\rangle ab^*$, one has:

$$d(\mathsf{E}_1) = \varepsilon \odot [\langle 6\rangle \odot \mathsf{1} \oplus \langle 10\rangle \odot a^* \oplus \langle 14\rangle \odot b^*] \oplus a \odot [\langle 11\rangle \odot b^*] \quad \text{(Example 3)}$$

$$d(a^*) = \varepsilon \odot [\langle 1\rangle \odot \mathsf{1}] \oplus a \odot [\langle 1\rangle \odot a^*] \qquad d(b^*) = \varepsilon \odot [\langle 1\rangle \odot \mathsf{1}] \oplus b \odot [\langle 1\rangle \odot b^*]$$

Therefore $d_\varepsilon(\mathsf{E}_1)$ is 6, and $d_\varepsilon(a^*) = d_\varepsilon(b^*) = 1$, from which $\mathcal{A}_{\mathsf{E}_1}$ follows: Fig. 1.

Example 5. The derived-term automaton of $((\langle\frac{1}{2}\rangle ab) \setminus (ab^*))^*$ is as follows. It has a non coaccessible state with a spontaneous loop whose weight, 1, is not starrable. This automaton must be trimmed to be valid.



Theorem 1. *For any expression* E, $\mathcal{A}_\mathsf{E}$ *is finite.*

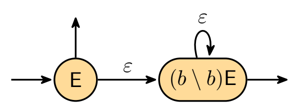
Proof. The proof goes in several steps (see Appendix A.5). First introduce the *proper derived terms* of E, a set of expressions noted $\mathrm{PD}(\mathsf{E})$, and the *derived terms* of E, $\mathrm{D}(\mathsf{E}) \coloneqq \mathrm{PD}(\mathsf{E}) \cup \{\mathsf{E}\}$. $\mathrm{PD}(\mathsf{E})$ admits a simple inductive definition similar to [2, Def. 3], to which we add $\mathrm{PD}(\mathsf{E} \setminus \mathsf{F}) \coloneqq \{\mathsf{E}' \setminus \mathsf{F}' \mid \mathsf{E}' \in \mathrm{PD}(\mathsf{E}), \mathsf{F}' \in \mathrm{PD}(\mathsf{F})\}$. Second, verify that $\mathrm{PD}(\mathsf{E})$ is finite. Third, prove that $\mathrm{D}(\mathsf{E})$ is "stable by expansion", i.e., $\forall \mathsf{F} \in \mathrm{D}(\mathsf{E}), \mathsf{exprs}\big(d(\mathsf{F})\big) \subseteq \mathrm{D}(\mathsf{E})$. Finally, observe that the states of $\mathcal{A}_\mathsf{E}$ are therefore members of $\mathrm{D}(\mathsf{E})$. □

Theorem 2. *If valid, any expression* E *and its expansion-based derived-term automaton* $\mathcal{A}_\mathsf{E}$ *denote the same series, i.e.,* $[\![\mathcal{A}_\mathsf{E}]\!] = [\![\mathsf{E}]\!]$.

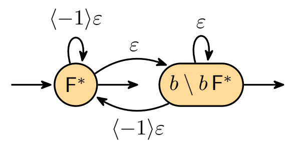
Proof. We show that the semantics of the states of $\mathcal{A}_\mathsf{E}$ (3) and of the expressions in $\mathrm{D}(\mathsf{E})$ define the same system of linear equations (Appendix A.6). □

The constant term of expressions without quotient can be computed syntactically [7, Definition 8], thus invalid expressions can be rejected during the construction of the derived-term automaton (when computing $d_\$(\mathsf{E})^*$ in (6)). This is no longer true with the quotient operator: the procedure may succeed on invalid expressions, the validity of the automaton [14] must be verified at end. The elimination of the spontaneous transitions is a means to check the validity of the automaton, but the computations highly depend on the semiring.

Example 6. In $\mathbb{Q}$, $\mathsf{E} \coloneqq (ab \setminus ab)^*$ is invalid as $[\![ab \setminus ab]\!] = [\![\varepsilon]\!]$ whose constant-term, 1, is not starrable in $\mathbb{Q}$. Therefore its derived-term automaton is invalid in $\mathbb{Q}$. However they are valid in $\mathbb{B}$.



The procedure may also build invalid automata from valid expressions. Consider for instance $\mathsf{F} \coloneqq ab \setminus ab + \langle -1 \rangle \mathsf{1}$: clearly $[\![\mathsf{F}]\!] = \mathsf{0}$, so $[\![\mathsf{F}^*]\!] = 1$. However the derived-term automaton of F^* is invalid: it has spontaneous loops whose weights are not starrable. This cannot happen in positive semirings.



5 Transposition and Right Quotient

This section introduces the support for the right quotient. We build it on top of a transpose operator, which might be used eventually with other operators.

Transpose. The *transpose* (aka *reversal* or *mirror image*) of a word $m = a_1 a_2 \dots a_n$ is $m^t \coloneqq a_n a_{n-1} \dots a_1$. The transpose of a series s is $s^t \coloneqq m \mapsto s(m^t)$.

Proposition 7. *For series $s, t \in \mathbb{K}\langle\!\langle A^* \rangle\!\rangle$:*

$$(s+t)^t = s^t + t^t \quad (ks)^t = k(s^t) \quad (sk)^t = (s^t)k \quad (st)^t = t^t s^t \quad s^{t^t} = s$$

Right quotient. We define the *right quotient* of two series s by t as $s \,/\, t \coloneqq v \mapsto \sum_{u \in A^*} s(vu) \cdot t(u)$. Since $\mathbb{K}$ is commutative, quotients are dual (see Appendix A.7).

Proposition 8. *If $\mathbb{K}$ is commutative, then $s \,/\, t = (t^t \setminus s^t)^t \qquad s \setminus t = (t^t \,/\, s^t)^t$.*

We extend Definition 1 with: $\mathsf{E} ::= \mathsf{0} \mid \mathsf{1} \mid a \mid \mathsf{E}+\mathsf{E} \mid \langle k \rangle \mathsf{E} \mid \mathsf{E} \cdot \mathsf{E} \mid \mathsf{E}^* \mid \mathsf{E} \setminus \mathsf{E} \mid \mathsf{E}^t$, with additional identities $\mathsf{0}^t \Rightarrow \mathsf{0}, \ell^t \Rightarrow \ell$ and we add $[\![\mathsf{E}^t]\!] \coloneqq [\![\mathsf{E}]\!]^t$ to Definition 2. Thanks to Proposition 8, we may add support for the right quotient as syntactic sugar on top of transposition and left quotient: $\mathsf{E} \,/\, \mathsf{F} \coloneqq (\mathsf{F}^t \setminus \mathsf{E}^t)^t$.

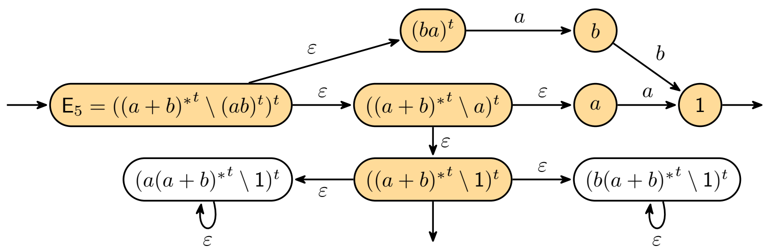
Definition 9. *The* transposed expansion of an expression E, *written $d^t(\mathsf{E})$, is defined inductively as follows:*

$$d^t(\mathsf{0}) \coloneqq d(\mathsf{0}) \qquad d^t(\mathsf{1}) \coloneqq d(\mathsf{1}) \qquad d^t(a) \coloneqq d(a)$$
$$d^t(\mathsf{E}+\mathsf{F}) \coloneqq d^t(\mathsf{E}) \oplus d^t(\mathsf{F}) \qquad d^t(\langle k \rangle \mathsf{E}) \coloneqq \langle k \rangle d^t(\mathsf{E})$$
$$d^t(\mathsf{E} \cdot \mathsf{F}) \coloneqq d^t_p(\mathsf{F}) \cdot \mathsf{E}^t \oplus \left\langle d^t_\$(\mathsf{F}) \right\rangle d^t(\mathsf{E}) \quad d^t(\mathsf{E}^*) \coloneqq \left\langle d^t_\$(\mathsf{E})^* \right\rangle \oplus \left\langle d^t_\$(\mathsf{E})^* \right\rangle d^t_p(\mathsf{E}) \cdot \mathsf{E}^{*t}$$
$$d^t(\mathsf{E} \setminus \mathsf{F}) \coloneqq d^t(\mathsf{E}) \setminus d^t(\mathsf{F}) \qquad d^t(\mathsf{E}^t) \coloneqq d(\mathsf{E})$$

where $d^t_\$(\mathsf{E})$ and $d^t_p(\mathsf{E})$ are the immediate constant term/immediate proper part of $d^t(\mathsf{E})$. Then Definition 7 is extended with $d(\mathsf{E}^t) \coloneqq d^t(\mathsf{E})$.

Proposition 6 is generalized by proving $\llbracket d^t(\mathsf{E}) \rrbracket = \llbracket \mathsf{E} \rrbracket^t$ (Appendix A.4).

Example 7. It is well known that the prefix of a language can be defined with $\mathrm{Pref}(\mathsf{E}) \coloneqq \mathsf{E} \mathbin{/} A^*$. Let $\mathsf{E}_5 \coloneqq (ab) \mathbin{/} (a+b)^* = ((a+b)^{*t} \setminus (ab)^t)^t$. We have $d(\mathsf{E}_5) = \varepsilon \odot [(ba)^t \oplus ((a+b)^{*t} \setminus a)^t]$. Its derived-term automaton is:



6 Related Work

The quotient between rational series is surprisingly little treated in the literature. Even Sakarovitch [15] defines the quotient by a word only: Sec. 1.2.3 p. 62 for the quotient of a word and of a language, and Sec. 4.1.1 p. 438 for the quotient of a series. It is quite rare to find the definition of the quotient of languages, and to define the quotient of *series* seems a unique feature of Li et al. [12][3].

Expansions were previously introduced [7] to optimize the construction of the derived-term automaton [13], and to add additional operators (the Hadamard product and complement). It was shown that they can also support multitape expressions [6]. Expansions previously appeared as an orphan concept from Brzozowski [4, last line of p. 484], and as "linear forms" by Antimirov [3, Def. 2.3].

For basic (weighted) expressions, there are more efficient algorithms to build the derived-term automaton [1, 5], but it is unclear how they could be extended to support operators such quotients. Actually, it is also doubtful whether the derivative-based approach [13] could be generalized to quotient, as the possible presence of ε in the firsts would correspond to derivatives with respect to ε.

Being able to feature ε in the firsts of expansions is a key feature. Indeed, Thiemann [16] shows that quotients have bad properties, in particular, they are not *ε-testable*. We avoided these issues by constructing an automaton with spontaneous transitions, which allows us to "delay" the computation of the constant-term of $a \setminus ab^*$ to the one of b^*. Besides, although transpose is neither left nor right derivable Thiemann [16], our procedure succeeds thanks to the introduction of the transposed computation of the expansion: d^t.

[3] When lifting the quotient of a language (or series) by a word to a quotient of languages, there are two options: *union* vs. *intersection* of the quotients by words. Li et al. [12] name *quotient* the union-based versions and write $s^{-1}t$ and st^{-1}, and name *residual* the intersection-based ones, written $s \setminus t$ and $s \mathbin{/} t$. In this paper, we focus only on left and right quotients, but denoted $s \setminus t$ and $s \mathbin{/} t$.

7 Conclusion

Thiemann [16] reported that the quotient and transpose operators pose real problems to the derivative-based construction of the derived-term automaton. We have addressed these issues in different ways. First, we rely on expansions rather than on derivatives, which allows us to cope naturally with spontaneous transitions, something that would correspond to nonsensical derivatives wrt the empty word. Second, since we can no longer determine the validity of an expression by a simple inductive computation, it is actually the validity of the derived-term automaton that ensures it. Finally, we introduce the transposed computation of expansions to handle the transpose operator.

In the future we will study the *residuals*, which, in the case of languages, rely on the intersection of quotients of words, rather than their union. We also want to explore other definitions of quotients, so that $\langle 2\rangle a \setminus \langle 2\rangle ab = a$, not $\langle 4\rangle a$.

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A Proofs

A.1 Proof of Proposition 1

This proof goes in several steps, with different constraints over s and t. From a formal point of view, it is actually "trivial": a simple look at the proof of Sakarovitch [15, Proposition III.2.6] shows that both expressions are *formally* equivalent. The real technical difficulty is semantic: ensuring that all the (infinite) sums are properly defined.

We actually only need Item 4 to establish Proposition 2.

1. *When s and t are proper.* This is a well-known consequence of Arden's lemma [15, Proposition III.2.5].
2. *When $s \in \mathbb{K}$, and t is proper.* This property holds when $\mathbb{K}$ is a strong topological semiring, and when s^* is defined [15, Proposition III.2.6].
3. *When $s, t \in \mathbb{K}$.* This result follows directly from the hypothesis of this property. Note however that $s^*(ts^*)^* = (s+t)^*$ is verified in all the "usual" semirings.
   - If $\mathbb{K}$ is a "usual numerical semiring" (i.e., $\mathbb{Q}, \mathbb{R}$, or more generally, a subring of $\mathbb{C}^n$), then s^* is the inverse of $1-s$, i.e., $(1-s)s^* = s^*(1-s) = 1$. To establish the result, we show that $s^*(ts^*)^*$ is the inverse of $1-(s+t)$. By hypothesis, s^* and $(ts^*)^*$ are defined. $(1-(s+t))s^*(ts^*)^* = (1-s)s^*(ts^*)^* - ts^*(ts^*)^* = (ts^*)^* - ts^*(ts^*)^* = (1-ts^*)(ts^*)^* = 1$, which shows that $(s+t)^*$ is defined.
   - If $\mathbb{K}$ is a tropical semiring, say, $\langle \mathbb{Z} \cup \{\infty\}, \min, +, \infty, 0 \rangle$, then s^* is defined iff $s \geq 0$, and then $s^* = 0$, hence the result trivially follows.
   - If $\mathbb{K}$ is the Log semiring, $\langle \mathbb{R}^+ \cup \{\infty\}, +_{\mathrm{Log}}, +, \infty, 0 \rangle$ where $+_{\mathrm{Log}} \coloneqq x, y \mapsto -\log(\exp(-x) + \exp(-y))$. Then we get $x^* = \log(1-\exp(-x))$. Again, one can verify the identity.
4. *When $s \in \mathbb{K}$ and t is any series.* By hypothesis, $(ts^*)^*$ is defined, i.e., $(t_\varepsilon s^*)^*$ is defined, so by Item 3, $(s+t_\varepsilon)^*$ is defined.

$$
\begin{aligned}
(s+t)^* &= (s+t_\varepsilon+t_p)^* \\
&= (s+t_\varepsilon)^*(t_p(s+t_\varepsilon)^*)^* && \text{by Item 2, } t_p \text{ proper, } (s+t_\varepsilon)^* \text{ defined} \\
&= s^*(t_\varepsilon s^*)^*(t_p s^*(t_\varepsilon s^*)^*)^* && \text{by Item 3} \\
&= s^*(t_\varepsilon s^* + t_p s^*)^* && \text{by Item 2, } t_p s^* \text{ proper, } (t_\varepsilon s^*)^* \text{ defined}
\end{aligned}
$$

$$
\begin{aligned}
&= s^*((t_\varepsilon + t_p)s^*)^* \\
&= s^*(ts^*)^*
\end{aligned}
$$

5. *When s is any series and t is proper.* By hypothesis, s^* is defined, so s_ε^* is defined.

$$
\begin{aligned}
(s+t)^* &= (s_\varepsilon + (s_p + t))* \\
&= s_\varepsilon^*((s_p+t)s_\varepsilon^*)^* && \text{by Item 2, } s_p + t \text{ proper} \\
&= s_\varepsilon^*(s_p s_\varepsilon^* + t s_\varepsilon^*)^* \\
&= s_\varepsilon^*(s_p s_\varepsilon^*)^*(t s_\varepsilon^*(s_p s_\varepsilon^*)^*)^* && \text{by Item 1, } s_p s_\varepsilon^* \text{ and } t s_\varepsilon^* \text{ are proper} \\
&= (s_\varepsilon + s_p)^*(t(s_\varepsilon + s_p)^*)^* && \text{by Item 2 } s_\varepsilon^* \text{ is defined, } s_p \text{ is proper} \\
&= s^*(ts^*)^*
\end{aligned}
$$

6. *When s and t are any series.* By hypothesis, s^* is defined.

$$
\begin{aligned}
(s+t)^* &= (s + t_\varepsilon + t_p)^* \\
&= (s+t_\varepsilon)^*(t_p(s+t_\varepsilon)^*)^* && \text{by Item 5, } t_p \text{ proper} \\
&= s^*(t_\varepsilon s^*)(t_p s^*(t_\varepsilon s^*)^*)^* && \text{by Item 4, } t_\varepsilon \in \mathbb{K} \\
&= s^*(t_\varepsilon s^* + t_p s^*)^* && \text{by by Item 5, } t_p s^* \text{ proper} \\
&= s^*(ts^*)^*
\end{aligned}
$$

A.2 Proof of Lemma 1

These are trivial consequences of the properties of the corresponding operations on series. For instance, let $\mathsf{P} = \bigoplus_{i\in[m]} \langle k_i \rangle \odot \mathsf{E}_i$, $\mathsf{Q} = \bigoplus_{j\in[n]} \langle h_j \rangle \odot \mathsf{F}_j$, we have:

$$
\begin{aligned}
[\![\mathsf{P} \setminus \mathsf{Q}]\!] &= \Big[\!\!\Big[\bigoplus_{i\in[m], j\in[n]} \big\langle k_i \cdot h_j \big\rangle \odot (\mathsf{E}_i \setminus \mathsf{F}_j) \Big]\!\!\Big] && \text{by definition} \\
&= \sum_{i\in[m], j\in[n]} \Big[\!\!\Big[\big\langle k_i \cdot h_j \big\rangle \odot (\mathsf{E}_i \setminus \mathsf{F}_j) \Big]\!\!\Big] \\
&= \sum_{i\in[m], j\in[n]} (k_i \cdot h_j) \cdot [\![\mathsf{E}_i \setminus \mathsf{F}_j]\!] \\
&= \sum_{i\in[m], j\in[n]} (k_i \cdot h_j) \cdot [\![\mathsf{E}_i]\!] \setminus [\![\mathsf{F}_j]\!] \\
&= \sum_{i\in[m], j\in[n]} (k_i \cdot [\![\mathsf{E}_i]\!]) \setminus (h_j \cdot [\![\mathsf{F}_j]\!]) && \text{by Proposition 3} \\
&= \sum_{i\in[m], j\in[n]} \Big[\!\!\Big[\langle k_i \rangle \odot \mathsf{E}_i \Big]\!\!\Big] \setminus \Big[\!\!\Big[\big\langle h_j \big\rangle \odot \mathsf{F}_j \Big]\!\!\Big] \\
&= \Big(\sum_{i\in[m]} \Big[\!\!\Big[\langle k_i \rangle \odot \mathsf{E}_i \Big]\!\!\Big] \Big) \setminus \Big(\sum_{j\in[n]} \Big[\!\!\Big[\big\langle h_j \big\rangle \odot \mathsf{F}_j \Big]\!\!\Big] \Big) && \text{by Proposition 3}
\end{aligned}
$$

$$= \left[\!\!\left[\bigoplus_{i \in [m]} \langle k_i \rangle \odot \mathsf{E}_i \right]\!\!\right] \setminus \left[\!\!\left[\bigoplus_{j \in [n]} \langle h_j \rangle \odot \mathsf{F}_j \right]\!\!\right]$$
$$= [\![\mathsf{P}]\!] \setminus [\![\mathsf{Q}]\!]$$

A.3 Proof of Lemma 2

The proofs are straightforward: lift semantic equivalences, such as those of Propositions 3 and 4, to syntax.

We prove for instance the case of the left quotient. However, we will use (5) rather than (4) for two reasons: not only is the proof more compact, it is also more general as it provides support for expressions and automata whose labels are words (e.g., "*abcd*"), not just letters or ε. In that case, one can verify that $d(\text{“}ab\text{”} \setminus \text{“}abcd\text{”}) = \varepsilon \odot [\langle 1_{\mathbb{K}} \rangle \odot \text{“}cd\text{”}]$.

The proof is as follows.

$$\begin{aligned}
[\![\mathsf{X} \setminus \mathsf{Y}]\!] &= \left[\!\!\left[\bigoplus_{\substack{\ell \in f(\mathsf{X}), \ell' \in f(\mathsf{Y}) \\ p = r(\ell, \ell')}} \varepsilon \odot \Big[((p \setminus \ell) \cdot \mathsf{X}_\ell) \setminus ((p \setminus \ell') \cdot \mathsf{Y}_\ell) \Big] \right]\!\!\right] && \text{by (5)} \\
&= \sum_{\substack{\ell \in f(\mathsf{X}), \ell' \in f(\mathsf{Y}) \\ p = r(\ell, \ell')}} [\![((p \setminus \ell) \cdot \mathsf{X}_\ell) \setminus ((p \setminus \ell') \cdot \mathsf{Y}_\ell)]\!] && \text{by Lemma 2 on } \oplus \\
&= \sum_{\substack{\ell \in f(\mathsf{X}), \ell' \in f(\mathsf{Y}) \\ p = r(\ell, \ell')}} \Big((p \setminus \ell) \cdot [\![\mathsf{X}_\ell]\!] \Big) \setminus \Big((p \setminus \ell') \cdot [\![\mathsf{Y}_{\ell'}]\!] \Big) && \text{by Lemma 1} \\
&= \sum_{\ell \in f(\mathsf{X}), \ell' \in f(\mathsf{Y})} \ell \cdot [\![\mathsf{X}_\ell]\!] \setminus \ell' \cdot [\![\mathsf{Y}_{\ell'}]\!] && \text{by Proposition 4} \\
&= \sum_{\ell \in f(\mathsf{X}), \ell' \in f(\mathsf{Y})} [\![\ell \cdot \mathsf{X}_\ell]\!] \setminus [\![\ell' \cdot \mathsf{Y}_{\ell'}]\!] && \text{by Lemma 1} \\
&= \Big(\sum_{\ell \in f(\mathsf{X})} [\![\ell \cdot \mathsf{X}_\ell]\!] \Big) \setminus \Big(\sum_{\ell' \in f(\mathsf{Y})} [\![\ell' \cdot \mathsf{Y}_{\ell'}]\!] \Big) && \text{by Proposition 3} \\
&= \left[\!\!\left[\bigoplus_{\ell \in f(\mathsf{X})} \ell \odot \mathsf{X}_\ell \right]\!\!\right] \setminus \left[\!\!\left[\bigoplus_{\ell' \in f(\mathsf{Y})} \ell' \odot \mathsf{Y}_{\ell'} \right]\!\!\right] && \text{by Lemma 2} \\
&= [\![\mathsf{X}]\!] \setminus [\![\mathsf{Y}]\!]
\end{aligned}$$

A.4 Proof of Proposition 6

A simple induction on E proves $[\![d(\mathsf{E})]\!] = [\![\mathsf{E}]\!]$, see the details in Demaille [7]. To handle transpose, we add the following case:

$$\begin{aligned}
[\![d^t(\mathsf{EF})]\!] &= \left[\!\!\left[d^t_p(\mathsf{F}) \cdot \mathsf{E}^t \oplus \langle d^t_\$(\mathsf{F}) \rangle d^t(\mathsf{E}) \right]\!\!\right] && \text{by Definition 9} \\
&= [\![d^t_p(\mathsf{F})]\!] [\![\mathsf{E}]\!]^t + d^t_\$(\mathsf{F}) [\![d(\mathsf{E})]\!]^t && \text{by Definition 2 and } [\![\mathsf{E}^t]\!]
\end{aligned}$$

$$
\begin{aligned}
&= \left[\!\left[d^t_p(\mathsf{F}) \right]\!\right] [\![\mathsf{E}]\!]^t + d^t_\$(\mathsf{F})[\![\mathsf{E}]\!]^t && \text{by induction hypothesis}\\
&= \left[\!\left[d^t_p(\mathsf{F}) + d^t_\$(\mathsf{F}) \right]\!\right] [\![\mathsf{E}]\!]^t\\
&= [\![d^t(\mathsf{F})]\!] [\![\mathsf{E}]\!]^t\\
&= [\![\mathsf{F}]\!]^t [\![\mathsf{E}]\!]^t = ([\![\mathsf{E}]\!][\![\mathsf{F}]\!])^t = [\![\mathsf{EF}]\!]^t && \text{by Proposition 7}
\end{aligned}
$$

A.5 Proof of Theorem 1

This proof shares large parts with the corresponding proof in Demaille [8, Appendix C], itself being based on the work from Lombardy and Sakarovitch [13]. As in the former we introduce $\mathrm{PD}(\mathsf{E})$, the *proper derived terms* of E, rather than $\mathsf{TD}(\mathsf{E})$, the *true derived terms* of E, as in the latter.

We will manipulate sets of expressions. To simplify notations, operations on expressions are lifted additively on sets of expressions. For instance:

$$\{\mathsf{E}_i \mid i \in [n]\} \setminus \{\mathsf{F}_j \mid j \in [m]\} \coloneqq \{\mathsf{E}_i \setminus \mathsf{F}_j \mid i \in [n], j \in [m]\}$$

Definition 10 (Derived Terms). *Given an expression* E*, its* proper derived terms *is the set* $\mathrm{PD}(\mathsf{E})$ *defined as follows:*

$$
\begin{gathered}
\mathrm{PD}(\mathsf{0}) \coloneqq \emptyset \qquad \mathrm{PD}(\mathsf{1}) \coloneqq \{\mathsf{1}\} \qquad \mathrm{PD}(a) \coloneqq \{\mathsf{1}\} \quad \forall a \in A\\
\mathrm{PD}(\mathsf{E}+\mathsf{F}) \coloneqq \mathrm{PD}(\mathsf{E}) \cup \mathrm{PD}(\mathsf{F}) \qquad \mathrm{PD}(\langle k\rangle \mathsf{E}) \coloneqq \mathrm{PD}(\mathsf{E}) \quad \forall k \in \mathbb{K}\\
\mathrm{PD}(\mathsf{E}\cdot\mathsf{F}) \coloneqq \mathrm{PD}(\mathsf{E})\cdot\mathsf{F} \cup \mathrm{PD}(\mathsf{F}) \qquad \mathrm{PD}(\mathsf{E}^*) \coloneqq \mathrm{PD}(\mathsf{E})\cdot\mathsf{E}^*\\
\mathrm{PD}(\mathsf{E}\setminus\mathsf{F}) \coloneqq \mathrm{PD}(\mathsf{E}) \setminus \mathrm{PD}(\mathsf{F})
\end{gathered}
$$

The derived terms *of an expression* E *is* $\mathrm{D}(\mathsf{E}) \coloneqq \mathrm{PD}(\mathsf{E}) \cup \{\mathsf{E}\}$.

Lemma 3. *For any expression* E*,* $\mathrm{D}(\mathsf{E})$ *is finite.*

Proof. Follows from the finiteness of $\mathrm{PD}(\mathsf{E})$, which is a direct consequence from Definition 10: finiteness propagates during the induction. ◻

Lemma 4 (Proper Derived Terms and Single Expansion). *For any expression* E*,* $\mathsf{exprs}\big(d(\mathsf{E})\big) \subseteq \mathrm{PD}(\mathsf{E})$.

Proof. Established by a simple verification of Definition 7. ◻

The derived terms of derived terms of E are derived terms of E. In other words, repeated expansions never “escape” the set of derived terms.

Lemma 5 (Proper Derived Terms and Repeated Expansions). *Let* E *be an expression. For all* $\mathsf{F} \in \mathrm{PD}(\mathsf{E})$*,* $\mathsf{exprs}\big(d(\mathsf{F})\big) \subseteq \mathrm{PD}(\mathsf{E})$.

Proof. This will be proved by induction over E.

Case $\mathsf{E} = \mathsf{0}$ or $\mathsf{E} = \mathsf{1}$. Trivially true, since there is no such F, as $\mathrm{PD}(\mathsf{E}) = \emptyset$.

Case $\mathsf{E} = a$. Then $\mathrm{PD}(\mathsf{E}) = \{\mathsf{1}\}$, hence $\mathsf{F} = \mathsf{1}$ and therefore $d(\mathsf{F}) = d(\mathsf{1}) = \langle 0_{\mathbb{K}} \rangle$, so $\mathsf{exprs}\big(d(\mathsf{F})\big) = \emptyset \subseteq \mathrm{PD}(\mathsf{E})$.

Case $\mathsf{E} = \mathsf{G} + \mathsf{H}$. Then $\mathrm{PD}(\mathsf{E}) = \mathrm{PD}(\mathsf{G}) \cup \mathrm{PD}(\mathsf{H})$. Suppose, without loss of generality, that $\mathsf{F} \in \mathrm{PD}(\mathsf{G})$. Then, by induction hypothesis, $\mathsf{exprs}\big(d(\mathsf{F})\big) \subseteq \mathrm{PD}(\mathsf{G}) \subseteq \mathrm{PD}(\mathsf{E})$.

Case $\mathsf{E} = \langle k \rangle \mathsf{G}$. Then if $\mathsf{F} \in \mathrm{PD}(\langle k \rangle \mathsf{G}) = \mathrm{PD}(\mathsf{G})$, so by induction hypothesis $\mathsf{exprs}\big(d(\mathsf{F})\big) \subseteq \mathrm{PD}(\mathsf{G}) = \mathrm{PD}(\langle k \rangle \mathsf{G}) = \mathrm{PD}(\mathsf{E})$.

Case $\mathsf{E} = \mathsf{G} \cdot \mathsf{H}$. Then $\mathrm{PD}(\mathsf{E}) = \{\mathsf{G}_i \cdot \mathsf{H} \mid \mathsf{G}_i \in \mathrm{PD}(\mathsf{G})\} \cup \mathrm{PD}(\mathsf{H})$.

- If $\mathsf{F} = \mathsf{G}_i \cdot \mathsf{H}$ with $\mathsf{G}_i \in \mathrm{PD}(\mathsf{G})$, then $d(\mathsf{F}) = d(\mathsf{G}_i \cdot \mathsf{H}) = d_p(\mathsf{G}_i) \cdot \mathsf{H} \oplus \big\langle d_\$(\mathsf{G}_i) \big\rangle d(\mathsf{H})$.
  Since $\mathsf{G}_i \in \mathrm{PD}(\mathsf{G})$ by induction hypothesis $\mathsf{exprs}\big(d_p(\mathsf{G}_i)\big) = \mathsf{exprs}\big(d(\mathsf{G}_i)\big) \subseteq \mathrm{PD}(\mathsf{G})$. By definition of the product of an expansion by an expression, $\mathsf{exprs}\big(d_p(\mathsf{G}_i) \cdot \mathsf{H}\big) \subseteq \{\mathsf{G}_j \cdot \mathsf{H} \mid \mathsf{G}_j \in \mathrm{PD}(\mathsf{G})\} \subseteq \mathrm{PD}(\mathsf{G} \cdot \mathsf{H}) = \mathrm{PD}(\mathsf{E})$.
- If $\mathsf{F} \in \mathrm{PD}(\mathsf{H})$, then by induction hypothesis $\mathsf{exprs}\big(d(\mathsf{F})\big) \subseteq \mathrm{PD}(\mathsf{H}) \subseteq \mathrm{PD}(\mathsf{E})$.

Case $\mathsf{E} = \mathsf{G}^*$. If $\mathsf{F} \in \mathrm{PD}(\mathsf{E}) = \{\mathsf{G}_i \cdot \mathsf{G}^* \mid \mathsf{G}_i \in \mathrm{PD}(\mathsf{G})\}$, i.e., if $\mathsf{F} = \mathsf{G}_i \cdot \mathsf{G}^*$ with $\mathsf{G}_i \in \mathrm{PD}(\mathsf{G})$, then $d(\mathsf{F}) = d(\mathsf{G}_i \cdot \mathsf{G}^*) = d_p(\mathsf{G}_i) \cdot \mathsf{G}^* \oplus \big\langle d_\$(\mathsf{G}_i) \big\rangle d(\mathsf{G}^*)$, so $\mathsf{exprs}\big(d(\mathsf{F})\big) \subseteq \mathsf{exprs}\big(d_p(\mathsf{G}_i) \cdot \mathsf{G}^*\big) \cup \mathsf{exprs}\big(d(\mathsf{G}^*)\big)$.[4] We will show that both are subsets of $\mathrm{PD}(\mathsf{E})$, which will prove the result.
Since $\mathsf{G}_i \in \mathrm{PD}(\mathsf{G})$, by induction hypothesis, $\mathsf{exprs}\big(d_p(\mathsf{G}_i)\big) = \mathsf{exprs}\big(d(\mathsf{G}_i)\big) \subseteq \mathrm{PD}(\mathsf{G})$, so by definition of a product of an expansion by an expression, $\mathsf{exprs}\big(d_p(\mathsf{G}_i) \cdot \mathsf{G}^*\big) \subseteq \{\mathsf{G}_j \cdot \mathsf{G}_j^* \mid \mathsf{G}_j \in \mathrm{PD}(\mathsf{G})\} = \mathrm{PD}(\mathsf{E})$.
By Lemma 4 $\mathsf{exprs}\big(d(\mathsf{G}^*)\big) \subseteq \mathrm{PD}(\mathsf{G}^*) = \mathrm{PD}(\mathsf{E})$.

Case $\mathsf{E} = \mathsf{G} \setminus \mathsf{H}$. (1) and (4) show that for all expansions X, Y,

$$\mathsf{exprs}\big(\mathsf{X} \setminus \mathsf{Y}\big) \subseteq \mathsf{exprs}(\mathsf{X}) \setminus \mathsf{exprs}(\mathsf{Y}) \tag{8}$$

Let $\mathsf{F} \in \mathrm{PD}(\mathsf{E}) = \mathrm{PD}(\mathsf{G}) \setminus \mathrm{PD}(\mathsf{H})$, i.e., let $\mathsf{F} = \mathsf{G}_i \setminus \mathsf{H}_j$ with $\mathsf{G}_i \in \mathrm{PD}(\mathsf{G}), \mathsf{H}_j \in \mathrm{PD}(\mathsf{H})$, then

$$\begin{aligned}
\mathsf{exprs}\big(d(\mathsf{F})\big) &= \mathsf{exprs}\big(d(\mathsf{G}_i \setminus \mathsf{H}_j)\big) \\
&= \mathsf{exprs}\big(d(\mathsf{G}_i) \setminus d(\mathsf{H}_j)\big) && \text{by (7)} \\
&\subseteq \mathsf{exprs}\big(d(\mathsf{G}_i)\big) \setminus \mathsf{exprs}\big(d(\mathsf{H}_j)\big) && \text{by (8)} \\
&\subseteq \mathrm{PD}(\mathsf{G}) \setminus \mathrm{PD}(\mathsf{H}) && \text{by induction hypothesis} \\
&= \mathrm{PD}(\mathsf{G} \setminus \mathsf{H}) && \text{by Definition 10} \\
&= \mathrm{PD}(\mathsf{E}) && \square
\end{aligned}$$

Lemma 6 (Derived Terms and Repeated Expansions). *Let E be an expression. For all $\mathsf{F} \in \mathrm{D}(\mathsf{E})$, $\mathsf{exprs}\big(d(\mathsf{F})\big) \subseteq \mathrm{PD}(\mathsf{E})$.*

Proof. Immediate consequence of Lemmas 4 and 5, since $\mathrm{D}(\mathsf{E}) = \mathrm{PD}(\mathsf{E}) \cup \{\mathsf{E}\}$. $\square$

[4] Given two expansions X, Y, $\mathsf{exprs}(\mathsf{X} \oplus \mathsf{Y}) \subseteq \mathsf{exprs}(\mathsf{X}) \cup \mathsf{exprs}(\mathsf{Y})$, but they may be different; consider for instance $\mathsf{X} = a \odot [\langle 1 \rangle \odot \mathsf{1}]$ and $\mathsf{Y} = a \odot [\langle -1 \rangle \odot \mathsf{1}]$ in $\mathbb{Z}$.

We may now prove Theorem 1.

Theorem 1 *For any expression* E, $\mathcal{A}_{\mathsf{E}}$ *is finite.*

Proof. The states of $\mathcal{A}_{\mathsf{E}}$ are members of $\mathrm{D}(\mathsf{E})$ (Lemma 6), which is finite (Lemma 3). □

A.6 Proof of Theorem 2

The Definition 8 shows that each state q_{F} of the $\mathcal{A}_{\mathsf{E}}$ has the following semantics:

$$[\![q_{\mathsf{F}}]\!] = \sum_{\substack{\ell \in f(d(\mathsf{F})) \\ \langle k \rangle \odot \mathsf{F}' \in d(\mathsf{F})(\ell)}} k_{\ell,\mathsf{F}'}\, \ell\, [\![q_{\mathsf{F}'}]\!] \tag{9}$$

Besides:

$$\begin{aligned}
[\![\mathsf{F}]\!] &= [\![d(\mathsf{F})]\!] && \text{(by Proposition 6)} \\
&= \Big[\!\!\Big[\bigoplus_{\ell \in f(d(\mathsf{F}))} \ell \odot d(\mathsf{F})(\ell) \Big]\!\!\Big] \quad = \sum_{\ell \in f(d(\mathsf{F}))} \ell [\![d(\mathsf{F})(\ell)]\!] \\
&= \sum_{\ell \in f(d(\mathsf{F}))} \ell \Big[\!\!\Big[\bigoplus_{\langle k_{\ell,i} \rangle \odot \mathsf{F}_{\ell,i} \in d(\mathsf{F})(\ell)} \langle k_{\ell,i} \rangle \odot \mathsf{F}_{\ell,i} \Big]\!\!\Big] \\
&= \sum_{\ell \in f(d(\mathsf{F}))} \ell \sum_{\langle k_{\ell,i} \rangle \odot \mathsf{F}_{\ell,i} \in d(\mathsf{F})(\ell)} k_{\ell,i} [\![\mathsf{F}_{\ell,i}]\!] \\
&= \sum_{\substack{\ell \in f(d(\mathsf{F})) \\ \langle k_{\ell,i} \rangle \odot \mathsf{F}_{\ell,i} \in d(\mathsf{F})(\ell)}} k_{\ell,i}\, \ell\, [\![\mathsf{F}_{\ell,i}]\!] && (10)
\end{aligned}$$

(9) and (10) define the same system of linear equations, hence $[\![\mathcal{A}_{\mathsf{E}}]\!] = [\![\mathsf{E}]\!]$. □

A.7 Proof of Proposition 8

$$\begin{aligned}
(t^t \setminus s^t)^t(v) &= (t^t \setminus s^t)(v^t) \\
&= \sum_{u \in A^*} t^t(v^t u) \cdot s^t(u) \\
&= \sum_{u \in A^*} t(u^t v) \cdot s(u^t) && \text{by definition of transpose} \\
&= \sum_{u \in A^*} t(uv) \cdot s(u) && \text{by change of variable: } u \to u^t \\
&= \sum_{u \in A^*} s(u) \cdot t(uv) && \text{by commutativity of } \mathbb{K} \\
&= (s \,/\, t)(v)
\end{aligned}$$

Commutativity may be replaced by a weaker condition: $\forall u, v \in A^*, t(uv) \cdot s(u) = s(u) \cdot t(uv)$.

The right-quotient is treated similarly.